General Relativity $\boxed{\text{Week 9}}$

Last time: We saw that the Einstein vacuum eqns $Ric_{\alpha\beta} = 0$ in wave coordinates $(g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0)$ become a quasilinear system of wave equations:

$$\Box_g g_{\mu\nu} = Q_{\mu\nu}(g, \partial g)$$

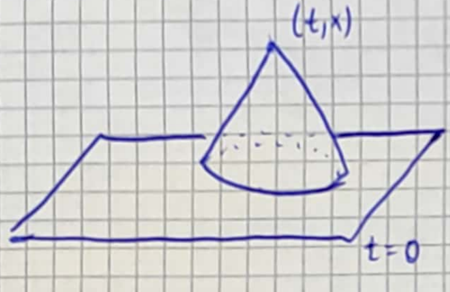
We want to understand the <u>initial value problem</u> for the Einstein eqns.

The simplest toy model:

$$\begin{cases} \Box_m \psi = 0 \text{ on } \mathbb{R}^{n+1} \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1 \end{cases} \qquad \begin{aligned} \Box_m &= -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2 \\ &= -\partial_t^2 + \Delta_{\mathbb{R}^n} \end{aligned}$$

It can be solved explicitly:

e.g. when $n=3$,
$$\psi(t,x) = \frac{1}{4\pi t^2} \int_{|y-x| = |t|} \left(t \cdot \psi_1(y) + \nabla\psi_0(y) \cdot (y-x) + \psi_0(y) \right) d\sigma_y$$



We can immediately deduce:
- Finite speed of propagation
- Domain of dependence:

  $\psi|_K$ depends only on initial data on $J^-(K) \cap \{t=0\}$

If we didn't know the explicit formula, how can we derive these properties and estimate the size of the solution?

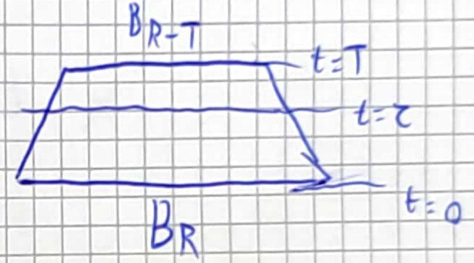
The fundamental energy identity:

Assume that ψ is in C^2, multiply with $\partial_t \psi$ $\longleftarrow$ ∂_t: timelike Killing

(algorithm related to Noether's theorem)

$$0 = -\Box_\eta \psi \cdot \partial_t \psi = \frac{1}{2}\partial_t\left((\partial_t\psi)^2\right) - \mathrm{div}_x\left(\partial_t\psi \cdot \nabla_x \psi\right) + \frac{1}{2}\partial_t\left(|\nabla_x \psi|^2\right)$$

Integrate over spacetime truncated cone:

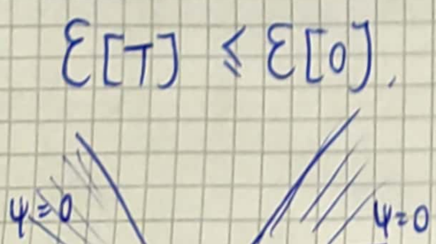


$$\Omega = \bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R-\tau}, \quad T \leq R.$$

Then
$$0 = \int_\Omega (-\Box_\eta \psi \cdot \partial_t \psi)\, dx\, dt = \frac{1}{2}\int_{\{t=T\}\cap B_{R-T}} \left((\partial_t\psi)^2 + |\nabla_x\psi|^2\right)dx - \frac{1}{2}\int_{\{t=0\}\cap B_R} (\partial_t\psi)^2 + |\nabla_x\psi|^2\, dx$$
$$- \int_0^T \underbrace{\int_{\{t=\tau\}\cap S_{R-\tau}} \left(\frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}|\nabla_x\psi|^2 - \partial_t\psi \cdot N\psi\right) d\sigma}_{\geq 0 \text{ by Cauchy-Schwarz!}}$$

So if $\mathcal{E}[\tau] = \frac{1}{2}\int_{\{t=\tau\}\cap B_{R-\tau}} \left((\partial_t\psi)^2 + |\nabla_x\psi|^2\right)dx$

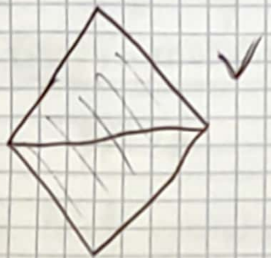
$\mathcal{E}[T] \leq \mathcal{E}[0]$. We can infer:
- Domain of dependence
- Uniqueness of solutions: If $(\psi, \partial_t\psi)|_{t=0} = 0 \Rightarrow \psi = 0$
- Propagation of compact support

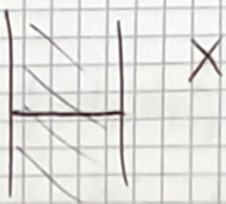


And crucially: Conservation of total energy for data of compact support.

We also need to study the well-posedness of the initial value problem: Existence, uniqueness and continuous dependence of the solution on the initial data.

On what domains is the initial value problem well-posed?





Answer: Globally hyperbolic!

More generally: <u>Covariant wave equation</u> (inhomogeneous) on (M, g):

$$\Box_g \psi = F \quad , \quad \Box_g \psi := g^{\mu\nu} \underbrace{(\nabla d\psi)_{\mu\nu}}_{\text{Hessian}} = \nabla^a \nabla_a \psi$$

$$= g^{\mu\nu} \partial_\mu \partial_\nu \psi - g^{\mu\nu} \Gamma^k_{\mu\nu} \partial_k \psi$$

$$= \frac{1}{\sqrt{-\det g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-\det g}\, \partial_\nu \psi \right)$$

When $g = \eta$: In Cartesian coordinates, the usual wave equation.

Initial value problem (Cauchy problem)

$$\begin{cases} \Box_g \psi = F & \text{on } M \\ \psi|_\Sigma = \psi_0 , \quad \hat{n}(\psi)|_\Sigma = \psi_1 \end{cases} \qquad (\hat{n}: \text{ timelike unit normal to } \Sigma)$$

We will show that for $(\psi_0, \psi_1) \in C^\infty \times C^\infty$, it is well-posed when Σ: <u>Cauchy</u>

Rmk: The cov. wave eqn $\Box_g \psi = 0$ comes from a Lagrangian (Euler-Lagrange eqn):

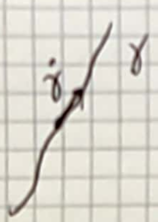
Formally, solutions are stationary points of

$$\int_M \mathcal{L}[\psi] \sqrt{-\det g}\, dx = \int_M g^{\mu\nu} \partial_\mu \psi\, \partial_\nu \psi \sqrt{-\det g}\, dx$$

The corresponding field theory provides us with the associated <u>energy-momentum</u> tensor:

$$T[\psi] = d\psi \otimes d\psi - \frac{1}{2} \langle d\psi, d\psi \rangle_g\, g$$

In local coordinates: $T_{\mu\nu}[\psi] = \partial_\mu \psi\, \partial_\nu \psi - \frac{1}{2} \partial^a \psi\, \partial_a \psi\, g_{\mu\nu}$



If γ is a timelike path traced by an observer:

$T_{\mu\nu} \dot\gamma^\mu \dot\gamma^\nu$ measures the energy density as measured by γ.

We will use T independently of its physical interpretation!

<u>Properties</u>: • If $\Box_g \psi = F \Rightarrow \nabla^\mu T_{\mu\nu} = F \cdot \partial_\nu \psi$

So if $\Box_g \psi = 0 \Rightarrow \nabla^\mu T_{\mu\nu} = 0$

(Proof: Use symmetry of Hessian: $(\nabla d\psi)_{a\beta} = (\nabla d\psi)_{\beta a}$)

• If $X, Y \in T_pM$ are <u>future directed</u> and <u>timelike</u>:

$\forall \psi \in C^1(M)$, $T_{\mu\nu}[\psi] X^\mu Y^\nu \geq c \cdot \sum_{a=0}^{n} (\partial_a \psi)^2$

c ↳ depends only on X, Y, g, coordinates

<u>Independent</u> of ψ!

<u>Proof</u>: • Easy if X, Y collinear (Choose coordinates where $X = \partial_0$)

• Else, if the 2-plane of X, Y is spanned by the fut. directed null vectors L, $\underline{L}$, normalized so that $\langle L, \underline{L} \rangle = -2$

$$X = aL + b\underline{L}$$
$$Y = cL + d\underline{L}, \quad a, b, c, d > 0$$

~~Complete~~ Let $\{e_2, \dots, e_n\}$ be orthonormal and orthogonal to $\{L, \underline{L}\}$, so that $\{L, \underline{L}, e_2, \dots, e_n\}$ is a double null frame.

Then:
$$T_{\mu\nu} L^\mu L^\nu = (L(\psi))^2$$
$$T_{\mu\nu} \underline{L}^\mu \underline{L}^\nu = (\underline{L}(\psi))^2$$
$$T_{\mu\nu} L^\mu \underline{L}^\nu = \sum_{i=1}^{n} (e_i(\psi))^2 \qquad \square$$

<u>Def</u>: Let X be a vector field on M. Current associated to X:

$$\underbrace{J^{(X)}_\mu[\psi]}_{\text{1-form}} := T_{\mu\nu}[\psi] X^\nu$$

Then:
$$\boxed{\nabla^\mu J^{(X)}_\mu = \Box\psi \cdot X(\psi) + T_{\mu\nu} \nabla^\mu X^\nu}$$

$$\frac{1}{2}(\nabla^\mu X^\nu + \nabla^\nu X^\mu) = {}^{(X)}\pi^{\mu\nu}$$

Deformation tensor:
$${}^{(X)}\pi_{\mu\nu} = \frac{1}{2}(\mathcal{L}_X g)_{\mu\nu}$$

<u>So</u>: If $\Box_g \psi = 0$ and X is <u>Killing</u>: $J^{(X)}_\mu$ is <u>conserved</u>! (Noether's theorem)

In the case of $\Box_\eta \psi = 0$: We esentially integrated the above identity for $X = \partial_t$ to obtain the energy identity.

We want to do something similar for the covariant wave eqn (on a spacetime without necessarily a timelike Killing v.f.)

<u>The divergence identity</u>:

- Recall that, on $\mathbb{R}^n$, with the Euclidean (Riemannian) metric:

$$\int_\Omega \operatorname{div} X = \int_{\partial\Omega} \langle \hat{n}, X \rangle$$

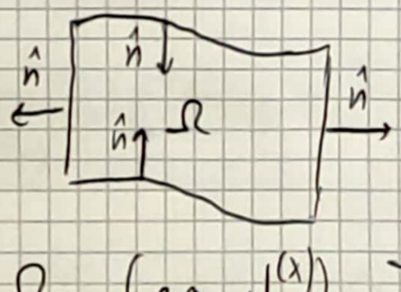
outward pointing!

- On a general manifold of dimension n: If ω is an $n-1$ form,

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega \qquad \text{(Stokes theorem)}$$

exterior derivative

<u>Prop</u>: Let $\Omega \subseteq (M, g)$ be a domain such that $\bar{\Omega}$ is compact and $\partial\Omega$ is piecewise smooth and having only spacelike and timelike components (i.e. not null).
Choose $\hat{n}$ according to:



Let ω be an 1-form on Ω (e.g. $J^{(X)}$). Then

$$\int_\Omega \operatorname{div} \omega \, dvol_g = \int_{\partial\Omega} \hat{n}^\mu \omega_\mu \, dvol_{\bar{g}}$$

$\longrightarrow$ Induced metric on $\bar{g}$

Rmk: • One can also treat the case when $\partial\Omega$ has null pieces (e.g the truncated cone in Minkowski)

In that case $\hat{n}$ and $dvol_{\partial\Omega}$ are not uniquely defined, but the "product" $\hat{n}\, dvol_{\partial\Omega}$ is.

(given a choice of (null) normal $\hat{n}$: define $dvol_{\partial\Omega}$ by $\underbrace{i_{\hat{n}}}_{\text{interior product}} \underbrace{dvol_g}_{n\text{-form}}$)

Proof:

Ω

Suppose (without loss of generality) that Ω lies inside a coordinate chart $(x^0, \dots, x^n)$ and is a coordinate cube, i.e. is of the form

$$\Omega = \bigcap_{a=0}^{n} \{ A^a < x^a < B^a \}.$$

Note: $dvol_g = \sqrt{-\det g}\, dx^0 \dots dx^n$

$$\operatorname{div} w = \nabla^\nu w_\nu = \frac{1}{\sqrt{-\det g}} \partial_a \left(g^{a\beta} \sqrt{-\det g}\, w_\beta \right)$$

So:

$$\int_\Omega \operatorname{div} w \, dvol_g = \int_{A^n}^{B^n} \cdots \int_{A^0}^{B^0} \frac{1}{\cancel{\sqrt{-\det g}}} \partial_a \left(g^{a\beta} \sqrt{-\det g}\, w_\beta \right) \cdot \cancel{\sqrt{-\det g}}\, dx^0 \dots dx^n$$

$$\overset{\text{Int.}}{=} \sum_{a=0}^{n} \left(\int_{(x^a = B^a) \cap \partial\Omega} g^{a\beta} \sqrt{-\det g}\, w_\beta \, dx^0 \dots \widehat{dx^a} \dots dx^n - \int_{x^a = A^a} \dots \right)$$

- For the normal to $\{x^a = \text{const}\}$:

It has to be parallel to $(dx^a)^{\#} = \nabla x^a$

$$\left((dx^a)^{\#}\right)^{\mu} = g^{\mu\nu}\partial_\nu x^a = g^{\mu a}$$

So $\left\{\begin{array}{l} \hat{n}^{\mu} = \lambda \cdot g^{\mu a} \\ \hat{n} \ \text{unit} \end{array}\right\} \quad \hat{n}^{\mu} = \pm \dfrac{1}{\sqrt{|g^{aa}|}} g^{a\mu}$

($g^{aa} > 0$: spacelike, $g^{aa} < 0$: timelike)

- If we denote with $\hat{g}_{(\alpha,\beta)}$ the matrix of g without the α row and β column:

$$g^{\alpha\beta} = (-1)^{\alpha+\beta}\frac{\det\left(\hat{g}_{(\alpha\beta)}\right)}{\det g}$$

So: $g^{aa} \cdot \det g = \det\left(\hat{g}_{(a,a)}\right) = \det \bar{g}_{\{x^a = \text{const}\}}$

↑ Induced metric on $\{x^a = \text{const}\}$.

Overall: $g^{a\beta}\omega_\beta\sqrt{-\det g} = \hat{n}^{\beta}\omega_\beta\sqrt{\pm\det\bar{g}}$ $\square$

<u>Remark</u>: Alternative proof using Stoke's theorem and the <u>Hodge star</u> operator: $\star \ \Lambda^k M \longrightarrow \Lambda^{n-k} M$

Since $\operatorname{div} = \pm \star d \star$, $d\mathrm{vol}_g = \star 1$, $\star\star = \pm 1$

$$\int_\Omega \operatorname{div}_g J \, d\mathrm{vol}_g = \pm\int_\Omega d\star J = \pm\int_{\partial\Omega}\star J \quad \ldots$$